Guidelines for Solving Problems of Particle Motion in Three Dimensions

1. Identify the problem:
    a. Number of degrees of freedoms
    b. Obvious constraints
    c. Intuitive notion of how system moves

2. Choose a suitable coordinate system
(Cartesian: x, y, z; Plane polar: r, θ; Cylindrical: R, ϕ, z; Spherical: r, θ, ϕ; etc.)
    a. Take advantage of system symmetry
    b. Take advantage of geometrical constraints

3. Find expressions for the components of the force in the chosen coordinate system:

Cartesian	*Plane Polar*	*Cylindrical*	*Spherical*
$F_x = \dots$	F_r	F_R	F_r
$F_y = \dots$	F_θ	F_ϕ	F_θ
$F_z = \dots$		F_z	F_ϕ

Examples:
- For 3D isotropic harmonic oscillator (Cartesian): $F_x = -kx; \quad F_y = -ky; \quad F_z = -kz.$
- For gravity (Spherical): $F_r = -\dfrac{GMm}{r^2}; \quad F_\theta = 0; \quad F_\phi = 0.$

4. Write the component equations of Newton's 2nd Law in the chosen coordinates:

Cartesian

$$m\ddot{x} = F_x \qquad m\ddot{y} = F_y \qquad m\ddot{z} = F_z$$

Plane Polar

$$m(\ddot{r} - r\dot{\theta}^2) = F_r$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$$

Cylindrical

$$m(\ddot{R} - R\dot{\phi}^2) = F_R$$
$$m(R\ddot{\phi} + 2\dot{R}\dot{\phi}) = F_\phi$$
$$m\ddot{z} = F_z$$

Spherical

$$m(\ddot{r} - r\dot{\phi}\sin^2\theta - r\dot{\theta}^2) = F_r$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) = F_\theta$$
$$m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta) = F_\phi$$

5. Make sure component equations are simplified as much as possible using any <u>constraints</u> for the motion of the system.

6. Check if force is conservative (i.e., check if $\nabla \times \vec{F} = 0$)

7. If possible, solve the component equations analytically. Otherwise, seek numerical solution.
    a. If variables separate, treat each component equation as a 1D problem
    b. If variables do not separate, attempt integration followed by substitution

8. If possible, find analytical expression for the particle path in space. Otherwise, obtain numerical plot of the path.

9. Compare your solution with original problem and make sure it makes sense.

<u>Other useful approaches to obtain expressions for the velocity:</u>
(Equivalent to first integration of equations of motion)

1. Linear momentum theorem: $m\vec{v} - m\vec{v}_0 = \int_{t_0}^{t} \vec{F} dt = \text{Impulse}$

2. Work-energy theorem: $\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r} = \text{Work by Net Force}$

3. Conservation of energy: $\frac{1}{2}mv^2 + V(\vec{r}) = E = \text{Total Mechanical Energy}$